

# Horizons, Constraints, and Black Hole Entropy

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**Abstract** Black hole entropy appears to be “universal”—many independent calculations, involving models with very different microscopic degrees of freedom, all yield the same density of states. I discuss the proposal that this universality comes from the behavior of the underlying symmetries of the classical theory. To impose the condition that a black hole be present, we must partially break the classical symmetries of general relativity, and the resulting Goldstone-boson-like degrees of freedom may account for the Bekenstein–Hawking entropy. In particular, I demonstrate that the imposition of a “stretched horizon” constraint modifies the algebra of symmetries at the horizon, allowing the use of standard conformal field theory techniques to determine the asymptotic density of states. The results reproduce the Bekenstein–Hawking entropy without any need for detailed assumptions about the microscopic theory.

## 1 Introduction

In the continuing quest for a quantum theory of gravity, black hole thermodynamics may be the nearest thing we have to “experimental” data. Hawking radiation has not been directly observed, of course. But the thermal properties of black holes have been derived in so many independent ways [1–9], and are so robust against changes in the starting assumptions [10, 11], that it would seem perverse to devote too much time to a putative quantum theory of gravity that could not reproduce the standard results.

Since the Bekenstein–Hawking entropy

$$S = \frac{A}{4\hbar G} \quad (1.1)$$

depends on both Planck’s constant and Newton’s constant, it is inherently quantum gravitational. We might therefore hope that black hole thermodynamics could give us important

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clues to the question of how to quantize general relativity. The fundamental problem is to understand the underlying statistical mechanics. What microscopic states are responsible for the Hawking temperature and Bekenstein–Hawking entropy?

Ten years ago, this question has an almost universally accepted answer: we don’t know. There were some interesting ideas floating around, involving entanglement entropy [12] and entropy of an “atmosphere” of external fields near the horizon [13], but we had nothing close to a complete description.

Today, things are radically different: many people will tell you with great confidence exactly what microscopic physics leads to black hole thermodynamics. The new problem is that while many people “know” the answer, they do not all agree. Black hole entropy may come from weakly coupled string and D-brane states [14, 15]; from nonsingular string “fuzzballs” [16]; from holographic states in a dual conformal field theory that is in some sense located at infinity [17]; from spin network states at the horizon [18] or perhaps inside the horizon [19]; from “heavy” degrees of freedom in induced gravity [20]; or perhaps from nonlocal topological properties of the black hole spacetime [21]. So far none of these pictures offers us a comprehensive picture of black hole thermodynamics. But in its realm of applicability, each can be used to count states for some black holes, and each seems to give the correct entropy (1.1).

In a field in which we do not yet know the answers, the existence of competing models may be seen as a sign of health. But the existence of competing models that all *agree* cries out for a deeper explanation.

## 2 Conformal Symmetry and State-Counting

While there may be others, I know of only one general approach that might explain this universality: perhaps some underlying feature of classical general relativity constrains the structure of any good quantum theory of gravity. The natural candidate for such a feature is a symmetry. This may at first seem unlikely—entropy is determined by the density of states, an inherently quantum mechanical characteristic, and it is not at all obvious that a classical symmetry can provide a strong enough constraint on the quantum theory. In one case, though, a known classical symmetry does just what we need.

Start with a two-dimensional conformal field theory, that is, a theory invariant under diffeomorphisms and Weyl transformations, and choose complex coordinates  $z$  and  $\bar{z}$ . The fundamental symmetries of such a theory are holomorphic and antiholomorphic diffeomorphisms, which are canonically generated by “Virasoro generators”  $L[\xi]$  and  $\bar{L}[\bar{\xi}]$  [22]. Such a theory has two conserved charges,  $L_0 = L[\xi_0]$  and  $\bar{L}_0 = \bar{L}[\bar{\xi}_0]$ , which can be thought of as “energies” with respect to constant holomorphic and antiholomorphic transformations, or equivalently as linear combinations of energy and angular momentum.

As generators of diffeomorphisms, the Virasoro generators have an algebra that is almost unique [23]:

$$\begin{aligned} \{L[\xi], L[\eta]\} &= L[\eta\xi' - \xi\eta'] + \frac{c}{48\pi} \int dz(\eta'\xi'' - \xi'\eta''), \\ \{L[\xi], \bar{L}[\bar{\eta}]\} &= 0, \\ \{\bar{L}[\bar{\xi}], \bar{L}[\bar{\eta}]\} &= \bar{L}[\bar{\eta}\bar{\xi}' - \bar{\xi}\bar{\eta}'] + \frac{\bar{c}}{48\pi} \int d\bar{z}(\bar{\eta}'\bar{\xi}'' - \bar{\xi}'\bar{\eta}''). \end{aligned} \tag{2.1}$$

The central charges (also known as “conformal anomalies”)  $c$  and  $\bar{c}$  determine the unique central extension of the ordinary algebra of diffeomorphisms. These constants can occur classically, coming, for instance, from boundary terms [24], and even if they are absent in the classical theory, they will typically appear upon quantization.

Consider now a conformal field theory for which the lowest eigenvalues of  $L_0$  and  $\bar{L}_0$  are nonnegative numbers  $\Delta_0$  and  $\bar{\Delta}_0$ . Cardy has shown [25, 26] that for large eigenvalues  $\Delta$  and  $\bar{\Delta}$  of  $L_0$  and  $\bar{L}_0$ , the density of states  $\rho(\Delta, \bar{\Delta})$  takes the remarkably simple form

$$\ln \rho(\Delta, \bar{\Delta}) \sim 2\pi \left\{ \sqrt{\frac{c_{\text{eff}}\Delta}{6}} + \sqrt{\frac{\bar{c}_{\text{eff}}\bar{\Delta}}{6}} \right\}, \quad \text{with } c_{\text{eff}} = c - 24\Delta_0, \quad \bar{c}_{\text{eff}} = \bar{c} - 24\bar{\Delta}_0. \quad (2.2)$$

The entropy is thus determined by the symmetry, independent of any other details. In particular, two conformal field theories with very different field content will have the same asymptotic density of states, provided that their effective central charges are equal.

The Cardy formula (2.2) is relatively straightforward to prove—see, for example, [27]—but I do not know of a fundamental physical explanation for this result. Partial insight may be obtained, though, by noting that a central charge reflects a breaking of the conformal symmetry. As we know from field theory [28], a broken symmetry can lead to new “Goldstone” degrees of freedom.<sup>1</sup> Here, for example, one would normally require that physical states be annihilated by the generators  $L[\xi]$  and  $\bar{L}[\bar{\xi}]$  of gauge symmetries,

$$L[\xi]|\text{phys}\rangle = \bar{L}[\bar{\xi}]|\text{phys}\rangle = 0. \quad (2.3)$$

But if  $c \neq 0$ , these conditions are incompatible with the brackets (2.1), and must be relaxed, for instance by requiring that only the positive frequency components of  $L[\xi]$  annihilate physical states. We thus obtain new physical states—“would-be gauge states” [30] that become physical because of the broken symmetry—that may contribute to  $\rho(\Delta, \bar{\Delta})$ .

### 3 Near-Horizon is Near-Conformal

General relativity is not a conformal field theory, and it is most certainly not two-dimensional. It is therefore not obvious that the Cardy formula should have any particular relevance to black hole entropy. But there are good reasons to believe that black hole dynamics may be *effectively* described by a two-dimensional conformal field theory near the horizon.

For instance, it is known that near a horizon, matter can be described by a two-dimensional conformal field theory, with fields depending only on  $t$  and the “tortoise coordinate”  $r_*$  [31–33]. Indeed, the near-horizon metric in such coordinates becomes, in any dimension,

$$ds^2 = N^2(dt^2 - dr_*^2) + ds_{\perp}^2, \quad (3.1)$$

where the lapse function  $N$  goes to zero at the horizon. The Klein–Gordon equation then reduces to

$$(\square - m^2)\varphi = \frac{1}{N^2}(\partial_t^2 - \partial_{r_*}^2)\varphi + \mathcal{O}(1) = 0. \quad (3.2)$$

<sup>1</sup>The analogy with the Goldstone mechanism was suggested to me by Kaloper and Terning [29].

The mass and transverse excitations become negligible near the horizon: they are essentially red-shifted away relative to excitations in the  $r_*, -t$  plane, leaving an effective two-dimensional conformal field theory at each point of the horizon. A similar dimensional reduction occurs for the Dirac equation.

Jacobson and Kang have observed that the surface gravity and temperature of a stationary black hole are conformally invariant as well [34]. And perhaps of most interest for the issues at hand, Medved et al. have shown that a generic stationary black hole metric always has an approximate conformal Killing vector near the horizon [35, 36].

#### 4 The BTZ Black Hole

The first solid evidence that the Cardy formula might be used to determine black hole thermodynamics came from the  $(2 + 1)$ -dimensional black hole of Bañados, Teitelboim, and Zanelli [37–40]. The BTZ black hole is the  $(2 + 1)$ -dimensional analog of the Kerr–AdS geometry, with a metric of the form

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\phi + N^\phi dt)^2$$

$$\text{with } N = \left( -8GM + \frac{r^2}{\ell^2} + \frac{16G^2 J^2}{r^2} \right)^{1/2}, \quad N^\phi = -\frac{4GJ}{r^2}, \quad (4.1)$$

where the cosmological constant is  $\Lambda = -1/\ell^2$  and  $M$  and  $J$  are the anti-de Sitter analogs of ADM mass and angular momentum. Like all vacuum spacetimes in  $2 + 1$  dimensions [41], the BTZ metric has constant curvature, and can in fact be expressed as a quotient of anti-de Sitter space by a discrete group of isometries. Nevertheless, it is a genuine black hole:

- It has an event horizon at  $r = r_+$  and an inner Cauchy horizon at  $r = r_-$ , where

$$r_\pm^2 = 4GM\ell^2 \left\{ 1 \pm \left[ 1 - \left( \frac{J}{M\ell} \right)^2 \right]^{1/2} \right\}; \quad (4.2)$$

- its Carter–Penrose diagram is identical to that of an ordinary Kerr–AdS black hole;
- it occurs as the end point of gravitational collapse of matter;
- most important for us, it exhibits standard black hole thermodynamics, with an entropy (1.1), where the two-dimensional “area” is the horizon circumference.

Despite its apparent simplicity, though, the thermodynamics of the BTZ black hole presents us with a conundrum. General relativity in  $2 + 1$  dimensions has no local degrees of freedom [41]: up to a finite number of global degrees of freedom, the metric is completely determined by the constraints. There seems to be no room for enough states to account for what can be an arbitrarily large entropy.

One piece of the answer was discovered independently in 1997 by Strominger [42] and Birmingham, Sachs, and Sen [43]. Note first that the conformal boundary of any  $(2 + 1)$ -dimensional asymptotically anti-de Sitter spacetime is a flat cylinder. It is thus unsurprising that the asymptotic symmetries of the BTZ black hole are described by a Virasoro algebra (2.1). It is a bit more surprising that this algebra has a central extension. But as Brown and Henneaux showed in 1986 [24]—and many authors have subsequently confirmed [40]—a classical central charge is in fact present, appearing because of the need to add boundary terms to the constraints, and taking the value

$$c = \frac{3\ell}{2\hbar G}. \quad (4.3)$$

Furthermore, the two conserved charges  $L_0$  and  $\bar{L}_0$  can be computed by standard ADM methods, yielding

$$\Delta_0 \sim \frac{1}{16\hbar G\ell} (r_+ + r_-)^2, \quad \bar{\Delta}_0 \sim \frac{1}{16\hbar G\ell} (r_+ - r_-)^2. \quad (4.4)$$

Inserting (4.3–4.4) into the Cardy formula, we find

$$S \sim \frac{2\pi}{8\hbar G} (r_+ + r_-) + \frac{2\pi}{8\hbar G} (r_+ - r_-) = \frac{2\pi r_+}{4\hbar G}, \quad (4.5)$$

the correct Bekenstein–Hawking entropy. We thus learn that the BTZ entropy is related to symmetries and boundary conditions at infinity.

A second piece of the answer starts from the observation [44, 45] that  $(2+1)$ -dimensional gravity with a negative cosmological constant can be written as an  $SO(2, 1) \times SO(2, 1)$  Chern–Simons theory. The Chern–Simons action is gauge invariant on a compact manifold. But boundaries and boundary conditions can break that invariance, in much the way I discussed in Sect. 2. The resulting Goldstone-like modes are described by a two-dimensional conformal field theory at infinity: a Wess–Zumino–Witten model [46–48], or, in the case of  $(2+1)$ -dimensional gravity, a Liouville theory with central charge (4.3) [49].

This result has now been confirmed by many different approaches, ranging from Chern–Simons methods to the AdS/CFT correspondence; see [40] for a review. Of particular interest is the explicit derivation of the Liouville field as a “would-be diffeomorphism” that becomes dynamical because of the need to impose boundary conditions [50, 51]. This appears to resolve our earlier paradox: although  $(2+1)$ -dimensional gravity on a compact manifold has only a small number of topological degrees of freedom, the presence of a boundary partially breaks the diffeomorphism invariance, promoting “gauge” degrees of freedom to physical excitations.

Whether these Liouville degrees of freedom can account for the entropy (4.5) remains an open question [40]. A naive application of the Cardy formula certainly yields the correct entropy. The relatively well-understood “normalizable” sector of Liouville theory has a nonzero minimum eigenvalue  $\Delta_0$  of  $L_0$ , however, lowering the effective central charge  $c_{\text{eff}}$  in (2.1) and ruining the correspondence [52, 53]. Chen has recently shown that the more poorly understood “nonnormalizable” sector of the theory may admit a sensible quantization, though, with states that can be explicitly counted and that seem to reproduce the correct entropy [54].

## 5 Horizons and Constraints

The real world, of course, is not  $2+1$  dimensional, and the derivation of the BTZ black hole entropy does not easily generalize. Only in  $2+1$  dimensions is the asymptotic boundary of spacetime two dimensional, allowing a direct application of the Cardy formula. Moreover, it seems more natural to associate black hole states with the horizon rather than spatial infinity. For a single black hole in  $2+1$  dimensions, the distinction may be unimportant, since there are no additional dynamical degrees of freedom lying between the horizon and infinity. In  $3+1$  dimensions, though, the prospect of extracting the degrees of freedom of a single black hole from the asymptotics of a complicated spacetime is daunting.

We can, however, draw a few lessons from  $2+1$  dimensions. We should look for “broken gauge invariance” from boundary conditions, and hope for an effective two-dimensional picture. But we should also start by looking near the horizon rather than at infinity.

There is an immediate objection to this proposal: unlike spatial infinity, the horizon of a black hole is not a true boundary. To clarify this issue, we must take a step back and ask what it means to ask a question about a black hole in a quantum theory of gravity.

In the usual semiclassical treatments of black hole thermodynamics, the answer to this question is obvious: we simply fix a black hole background, and look at quantum fields in that background. In a full quantum theory of gravity, though, we are not allowed to do this. Such a theory has no fixed background; the metric is an operator, and the uncertainty principle tells us that its value cannot be exactly specified. At best, we can restrict a piece of the geometry and ask *conditional* questions: “If geometric features that characterize a black hole of type  $X$  are present, what is the probability of observing phenomenon  $Y$ ?”

I know of two ways to obtain such a conditional probability. The first, discussed in [30], is to treat the horizon as a “boundary” at which suitable boundary conditions are imposed. In the sum over histories formalism, for example, we can divide spacetime into two regions along a hypersurface  $\mathcal{H}$  and perform separate path integrals over fields on each side, with fields restricted at the “boundary” by the requirement that  $\mathcal{H}$  be a horizon. Such split path integral has been studied in detail in  $2 + 1$  dimensions [55], where it leads to the same WZW model that was discussed in the preceding section. Although the horizon is not a true boundary, it is, in this approach, a hypersurface upon which we impose “boundary conditions”, and this turns out to be good enough.

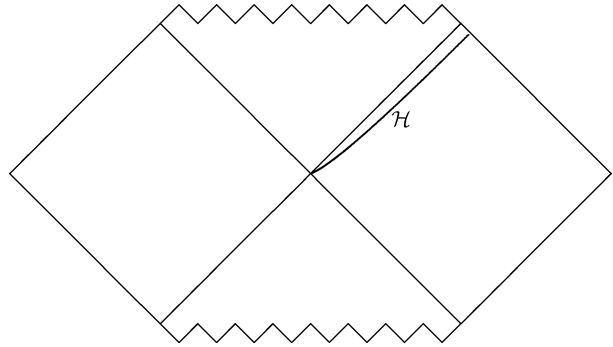
Alternatively, we can impose “horizon constraints” directly, either classically or in the quantum theory. We might, for example, construct an operator  $\vartheta$  representing the expansion of a particular null surface, and restrict ourselves to states annihilated by  $\vartheta$ . As we shall see below, such a restriction can affect the algebra of diffeomorphisms, allowing us to exploit the Cardy formula to count states.

The “horizon as a boundary” approach has been widely investigated; see, for instance, [56–65]. One naturally finds a conformal symmetry in the  $r_*$ - $t$  plane, and one can obtain a Virasoro algebra with a central charge that leads to the correct black hole entropy. On the other hand, the diffeomorphisms whose algebra yields that central charge—essentially those that leave the lapse function invariant—are generated by vector fields that blow up at the horizon [66–69], and it is not clear whether this is permissible. Nor is the angular dependence of these vector fields well understood. A related approach looks for approximate conformal symmetry near the horizon [70–73]; again, one finds a Virasoro algebra with a central charge that seems to lead to the correct entropy.

The alternative “horizon constraint” approach is still in the early stages of development. Suppose we wish to constrain our theory of gravity by requiring that some prescribed surface  $\mathcal{H}$  be an “isolated horizon” [74] or a “dynamical horizon” [75]. Such constraints restrict the allowed data on  $\mathcal{H}$ , and in principle we should be able to use the tools of constrained Hamiltonian dynamics [76–78] to study such conditions. Unfortunately, though, an isolated horizon is by definition a null surface, while standard techniques deal with constraints on spacelike surfaces. While some work on “constrained light cone quantization” exists, this is a difficult program.

As a simpler warm-up exercise, we can impose constraints requiring the presence of a spacelike “stretched horizon” that becomes nearly null, as illustrated in Fig. 1. As I will discuss below, such a stretched horizon constraint leads to a Virasoro algebra with a calculable central charge, and at least in the case of two-dimensional dilaton gravity, yields the correct Bekenstein–Hawking entropy.

**Fig. 1** A spacelike “stretched horizon”  $\mathcal{H}$



**6 Horizon Constraints and the Dilaton Black Hole**

I now turn to a slightly more technical analysis of a particular case, the two-dimensional dilaton black hole. Details can be found in [79, 80]. Our world is surely not two dimensional, so this may again seem too great a specialization. As argued above, though, we expect the near-horizon dynamics of an arbitrary black hole to be effectively two dimensional, so a dimensionally reduced model is not such a bad starting point.

Two-dimensional dilaton gravity [81, 82] can be described, after a rescaling of the metric, by an action

$$I = \int d^2x \sqrt{-g} [AR + V(A)], \tag{6.1}$$

where  $R$  is the two-dimensional scalar curvature and  $A$  is a scalar field, the dilaton (often denoted as  $\varphi$ ).  $V(A)$  is a potential whose form depends on the higher-dimensional theory we started with; we will not need an exact expression. As the notation suggests,  $A$  is the transverse area in the higher-dimensional theory, in units  $16\pi G = 1$ . The analog of the expansion—the fractional rate of change of area along a null curve with null normal  $l^a$ —is

$$\vartheta = l^a \nabla_a A / A. \tag{6.2}$$

It is useful to rewrite the action (6.1) in terms of a null dyad ( $l^a, n^a$ ) with  $l^2 = n^2 = 0, l \cdot n = -1$ . These determine “surface gravities”  $\kappa$  and  $\bar{\kappa}$ , defined by the conditions

$$\nabla_a l_b = -\kappa n_a l_b - \bar{\kappa} l_a l_b, \quad \nabla_a n_b = \kappa n_a n_b + \bar{\kappa} l_a n_b, \tag{6.3}$$

and the action becomes

$$I = \int d^2x [\hat{\epsilon}^{ab} (2\kappa n_b \partial_a A - 2\bar{\kappa} l_b \partial_a A) + \sqrt{-g} V]. \tag{6.4}$$

If we now express the components of our dyad with respect to coordinates  $(u, v)$  as

$$l = \sigma du + \alpha dv, \quad n = \beta du + \tau dv, \tag{6.5}$$

it is easy to find the Hamiltonian form of the action [79]. The system has three first-class constraints; denoting a derivative with respect to  $v$  by a prime, they are

$$\begin{aligned}
 C_{\perp} &= \pi_{\alpha'} - \frac{1}{2}\pi_{\alpha}\pi_A - \tau V(A), \\
 C_{\parallel} &= \pi_A A' - \alpha\pi_{\alpha'} - \tau\pi_{\tau}', \\
 C_{\pi} &= \tau\pi_{\tau} - \alpha\pi_{\alpha} + 2A'.
 \end{aligned}
 \tag{6.6}$$

$C_{\perp}$  and  $C_{\parallel}$  generate the diffeomorphisms orthogonal to and parallel to a spacelike slice  $u = \text{const.}$ , while  $C_{\pi}$  generates local Lorentz transformations.

We can now impose “stretched horizon” constraints at the surface  $u = 0$ . We first demand that  $\mathcal{H}$  be “almost null”, i.e., that its normal be nearly equal to the null vector  $l^a$ . By (6.5), this requires that  $\alpha = \epsilon_1 \ll 1$ .

We must next demand that  $\mathcal{H}$  be “almost nonexpanding”. This is a bit more subtle, since the absolute scale of  $l^a$  is not fixed; while the requirement that the expansion be exactly zero is independent of such a scale, the requirement of a “small” expansion is not. One solution is to note that the surface gravity  $\kappa$  depends on this scale as well, and that the ratio  $\vartheta/\kappa A$  is independent of at least constant rescalings of  $l^a$ . We therefore require that  $l^{\nu}\nabla_{\nu}A/\kappa A = \epsilon_2 \ll 1$ . Expressing these conditions in terms of canonical variables, we obtain two “stretched horizon” constraints:

$$\begin{aligned}
 K_1 &= \alpha - \epsilon_1 = 0, \\
 K_2 &= A' - \frac{1}{2}\epsilon_2 A_+ \pi_A + \frac{a}{2}C_{\pi} = 0,
 \end{aligned}
 \tag{6.7}$$

where  $a$  is an arbitrary constant and  $A_+$  is the horizon value of the dilaton. By looking at a generic exact black hole solution, one can verify that these constraints do, in fact, determine a spacelike stretched horizon that looks like that of Fig. 1, which very rapidly becomes very nearly null.

$K_1$  and  $K_2$  are not quite “constraints” in the usual sense of constrained Hamiltonian dynamics, but they are similar enough that many existing techniques can be used. In particular, observe that the  $K_i$  have nontrivial brackets with the momentum and boost generators  $C_{\parallel}$  and  $C_{\pi}$ , so these no longer generate invariances of the constrained theory. But we can fix this by a method suggested years ago by Bergmann and Komar [78]: we define new generators

$$\begin{aligned}
 C_{\parallel} &\rightarrow C_{\parallel}^* = C_{\parallel} + a_1 K_1 + a_2 K_2, \\
 C_{\pi} &\rightarrow C_{\pi}^* = C_{\pi} + b_1 K_1 + b_2 K_2
 \end{aligned}
 \tag{6.8}$$

with coefficients  $a_i$  and  $b_i$  chosen so that  $\{C^*, K_i\} = 0$ . Since  $K_i = 0$  on admissible geometries, the generators  $C^*$  are physically equivalent to the original  $C$ ; but they now preserve the horizon constraints as well.

We now make the crucial observation that the redefinitions (6.8) affect the Poisson brackets of the constraints. With the choice  $a = -2$  in (6.7), it may be shown that

$$\begin{aligned}
 \{C_{\parallel}^*[\xi], C_{\parallel}^*[\eta]\} &= -C_{\parallel}^*[\xi\eta' - \eta\xi'] + \frac{1}{2}\epsilon_2 A_+ \int dv(\xi'\eta'' - \eta'\xi''), \\
 \{C_{\parallel}^*[\xi], C_{\pi}^*[\eta]\} &= -C_{\pi}^*[\xi\eta'], \\
 \{C_{\pi}^*[\xi], C_{\pi}^*[\eta]\} &= -\frac{1}{2}\epsilon_2 A_+ \int dv(\xi\eta' - \eta\xi').
 \end{aligned}
 \tag{6.9}$$



This algebra has a simple conformal field theoretical interpretation [22]: the  $C_{\parallel}^*$  generate a Virasoro algebra with central charge

$$\frac{c}{48\pi} = -\frac{1}{2}\epsilon_2 A_+, \tag{6.10}$$

while  $C_{\pi}^*$  is a primary field of weight one. Different choices of the parameter  $a$  in (6.7) yield equivalent algebras, although with slightly redefined generators.

The Cardy formula (2.2) requires both the central charge and the conserved charge  $\Delta$ . As in the usual approaches to black hole mechanics, the latter comes from a boundary term needed to make the generator  $C_{\parallel}^*$  “differentiable” [83]. It was shown in [79] that this term is

$$C_{\parallel \text{bdry}}^*[\xi] = -\xi \pi_A A|_{v=v_+}, \tag{6.11}$$

which will give a nonvanishing classical contribution to  $\Delta$ .

Finally, we also need a mode expansion to define the Fourier component  $L_0$ , or, equivalently, a normalization for the “constant translation”  $\xi_0$ . For a conformal field theory defined on a circle, or on a full complex plane with a natural complex coordinate, this normalization is essentially unique. Here, though, it is not so obvious how to choose the “right” complex coordinate. As argued in [71], however, there is one particularly natural choice,

$$z = e^{2\pi i A/A_+}, \quad \xi_n = \frac{A_+}{2\pi A'} z^n, \tag{6.12}$$

where the prefactor is fixed by demanding that  $[\xi_m, \xi_n] = i(n - m)\xi_{m+n}$ .

Equation (6.11) then implies that

$$\Delta = C_{\parallel \text{bdry}}^*[\xi_0] = -\frac{A_+}{2\pi A'} \pi_A A_+ = -\frac{A_+}{\pi \epsilon_2}. \tag{6.13}$$

Inserting (6.10) and (6.13) into the Cardy formula, assuming that  $\Delta_0$  is small, and restoring the factors of  $16\pi G$  and  $\hbar$ , we obtain an entropy

$$S = \frac{2\pi}{16\pi G} \sqrt{\left(-\frac{24\pi \epsilon_2 A_+}{6\hbar}\right) \left(-\frac{A_+}{\pi \epsilon_2 \hbar}\right)} = \frac{A_+}{4\hbar G}, \tag{6.14}$$

exactly reproducing the standard Bekenstein–Hawking entropy (1.1).

### 7 Open Questions

While these results are intriguing, they are certainly not yet conclusive. I know of several straightforward steps that may take us further:

1. We should determine how sensitive the result is to the exact definition (6.7) of the stretched horizon  $\mathcal{H}$ . As a first step, a similar computation has now been carried out in radially quantized Euclidean quantum gravity [84]. Here, the constraints have a simpler geometric interpretation—they essentially fix the proper distance of the initial surface from the origin—and it may be possible to relate the constraint analysis to the path integral methods of [85]. Ideally, the analysis should also be repeated in a true light cone quantization; work on this is in progress.

2. We should extend the analysis beyond two dimensions. This is probably not too hard conceptually, though the technical details may be complicated.
3. We should try to relate the horizon constraint approach of Sect. 6 to the “horizon as boundary” methods described in Sect. 5. A comparison with the near-horizon symmetry approaches of [70, 71] should be fairly simple: the central charge (6.10) agrees with that of [71], and can be made to match that of [70] by a choice of the parameter  $q$  in that paper. The relation to other boundary approaches may be more subtle, though it is worth noting that the central charge (6.10) agrees with that of [56] with the interesting choice  $T = \vartheta$  for the periodicity of the modes in that paper.

Three other questions are more difficult, but perhaps more profound. First, observe that the vector fields (6.12) blow up at the horizon, where  $A' \rightarrow 0$ . A similar phenomenon occur in the “horizon as boundary” approach, as noted in [66, 68]. This divergence seems to be closely related to the use of Schwarzschild-like coordinates, which characterize an observer who stays outside of the black hole. Note that such coordinates were needed for the conformal behavior described in Sect. 3. Similarly, an exact horizon constraint has recently been analyzed in two-dimensional dilaton gravity, with diffeomorphisms that are implicitly required to be well-behaved at the horizon [86]; for such diffeomorphisms, the central extension of the Virasoro algebra seems to disappear. If this proves to be a general feature of conformal methods, it may be telling us something profound about “black hole complementarity” [87]: perhaps the Bekenstein–Hawking entropy is only well-defined for an observer who remains outside the horizon.

Second, if the horizon symmetry described here provides a universal explanation of black hole entropy, then the symmetry should be identifiable in other approaches to black hole microphysics. There is a reasonable chance that this connection can be made for a large class of “stringy” black holes. Many of the higher dimensional black holes whose entropy can be computed in string theory have near-horizon geometries that look like that of the BTZ black hole [88], allowing thermodynamic properties to be computed by the methods of Sect. 4. If the Virasoro algebra of the BTZ black hole at infinity can be related to our near-horizon algebra, it may be possible to demonstrate the role of horizon constraints in these string theoretical black holes. A similar interpretation may be possible for induced quantum gravity, where a conformal field theory description is also possible [89]. Whether a corresponding result exists for loop quantum gravity is an open question.

Third, there is much more to black hole thermodynamics than the Bekenstein–Hawking entropy. If the ideas described here are correct, the Goldstone-like “would-be diffeomorphism” degrees of freedom must couple properly to external fields to produce Hawking radiation. In  $2 + 1$  dimensions, there has been one computation of this sort [90], in which the coupling of a classical source to the boundary degrees of freedom was shown to yield the correct Hawking radiation. A recent discussion of Hawking radiation in terms of diffeomorphism anomalies at the horizon [91] may also be relevant, and it may be possible to obtain some information by imposing horizon constraints in a Lagrangian formalism [92]. But the investigation of this issue has barely begun.

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## References

1. Hawking, S.W.: Nature **248**, 30 (1974)

2. Gibbons, G.W., Hawking, S.W.: *Phys. Rev. D* **15**, 2752 (1977)
3. Gibbons, G.W., Perry, M.J.: *Proc. Roy. Soc. Lond. A* **358**, 467 (1978)
4. Fredenhagen, K., Haag, R.: *Commun. Math. Phys.* **127**, 273 (1990)
5. Kay, B.S., Wald, R.M.: *Phys. Rep.* **207**, 49 (1991)
6. Unruh, W.G.: *Phys. Rev. D* **14**, 870 (1976)
7. DeWitt, B.S.: In: Hawing, S.W., Israel, W.W. (eds.) *General Relativity: an Einstein Centenary Survey*, Cambridge University Press, Cambridge (1979), Chap. 14
8. Parikh, M.K., Wilczek, F.: *Phys. Rev. Lett.* **85**, 5042 (2000), hep-th/9907001
9. Nadalini, N., Vanzo, L., Zerbini, S.: hep-th/0511250
10. Barceló, C., Liberati, S., Visser, M.: *Living Rev. Relativ.* **8**, 12 (2005)
11. Unruh, W.G., Schützhold, R.: *Phys. Rev. D* **71**, 024028 (2005), gr-qc/0408009
12. Bombelli, L., Koul, R.K., Lee, J., Sorkin, R.: *Phys. Rev. D* **34**, 373 (1986)
13. 't Hooft, G.: *Nucl. Phys. B* **256**, 727 (1985)
14. Strominger, A., Vafa, C.: *Phys. Lett. B* **379**, 99 (1996), hep-th/9601029
15. Peet, A.W.: *Class. Quantum Gravity* **15**, 3291 (1998), hep-th/9712253
16. Mathur, S.D.: *Fortschr. Phys.* **53**, 793 (2005), hep-th/0502050
17. Aharony, O., Gubser, S.S., Maldacena, J.M., Ooguri, H., Oz, Y.: *Phys. Rep.* **323**, 183 (2000), hep-th/9905111
18. Ashtekar, A., Baez, J., Corichi, A., Krasnov, K.: *Phys. Rev. Lett.* **80**, 904 (1998), gr-qc/9710007
19. Livine, E.R., Terno, D.R.: gr-qc/0508085
20. Frolov, V.P., Fursaev, D.V.: *Phys. Rev. D* **56**, 2212 (1997), hep-th/9703178
21. Hawking, S.W., Hunter, C.J.: *Phys. Rev. D* **59**, 044025 (1999), hep-th/9808085
22. Di Francesco, P., Mathieu, P., Sénéchal, D.: *Conformal Field Theory*. Springer, New York (1997)
23. Teitelboim, C.: In: Christensen, S.M. (ed.) *Quantum Theory of Gravity*. Hilger, Bristol (1984)
24. Brown, J.D., Henneaux, M.: *Commun. Math. Phys.* **104**, 207 (1986)
25. Cardy, J.A.: *Nucl. Phys. B* **270**, 186 (1986)
26. Blöte, H.W.J., Cardy, J.A., Nightingale, M.P.: *Phys. Rev. Lett.* **56**, 742 (1986)
27. Carlip, S.: *Class. Quantum Gravity* **15**, 3609 (1998), hep-th/9806026
28. Coleman, S.: *Aspects of Symmetry*. Cambridge University Press, Cambridge (1985), Chap. 5
29. Kaloper, N., Terning, J.T.: personal communication (2004)
30. Carlip, S.: In: de Alfaro, V., et al. (eds.) *Constrained Dynamics and Quantum Gravity 1996*. *Nucl. Phys. Proc. Suppl.* **57**. Elsevier, Amsterdam (1997), gr-qc/9702017
31. Birmingham, D., Gupta, K.S., Sen, S.: *Phys. Lett. B* **505**, 191 (2001), hep-th/0102051
32. Gupta, K.S., Sen, S.: *Phys. Lett. B* **526**, 121 (2002), hep-th/0112041
33. Camblong, H.E., Ordóñez, C.R.: *Phys. Rev. D* **71**, 104029 (2005), hep-th/0411008
34. Jacobson, T., Kang, G.: *Class. Quantum Gravity* **10**, L201 (1993), gr-qc/9307002
35. Medved, A.J.M., Martin, D., Visser, M.: *Class. Quantum Gravity* **21**, 3111 (2004), gr-qc/0402069
36. Medved, A.J.M., Martin, D., Visser, M.: *Phys. Rev. D* **70**, 024009 (2004), gr-qc/0403026
37. Bañados, M., Teitelboim, C., Zanelli, J.: *Phys. Rev. Lett.* **69**, 1849 (1992), hep-th/9204099
38. Bañados, M., Henneaux, M., Teitelboim, C., Zanelli, J.: *Phys. Rev. D* **48**, 1506 (1993), gr-qc/9302012
39. Carlip, S.: *Class. Quantum Gravity* **12**, 2853 (1995), gr-qc/9506079
40. Carlip, S.: *Class. Quantum Gravity* **22**, R85 (2005), gr-qc/0503022
41. Carlip, S.: *Living Rev. Relativ.* **8**, 1 (2005), gr-qc/0409039
42. Strominger, A.: *J. High Energy Phys.* **02**, 009 (1998), hep-th/9712251
43. Birmingham, D., Sachs, I., Sen, S.: *Phys. Lett. B* **424**, 275 (1998), hep-th/9801019
44. Achúcarro, A., Townsend, P.K.: *Phys. Lett. B* **180**, 89 (1986)
45. Witten, E.: *Nucl. Phys. B* **311**, 46 (1988)
46. Witten, E.: *Commun. Math. Phys.* **92**, 455 (1984)
47. Elitzur, S., Moore, G.W., Schwimmer, A., Seiberg, N.: *Nucl. Phys. B* **326**, 108 (1989)
48. Carlip, S.: *Nucl. Phys. B* **362**, 111 (1991)
49. Coussaert, O., Henneaux, M., van Driel, P.: *Class. Quantum Gravity* **12**, 2961 (1995), gr-qc/9506019
50. Manvelyan, R., Mkrtychyan, R., Müller-Kirsten, H.J.W.: *Phys. Lett. B* **509**, 143 (2001), hep-th/0103082
51. Carlip, S.: *Class. Quantum Gravity* **22**, 3055 (2005), gr-qc/0501033
52. Kutasov, D., Seiberg, N.: *Nucl. Phys. B* **358**, 600 (1991)
53. Martinec, E.J.: hep-th/9809021
54. Chen, Y.-J.: *Class. Quantum Gravity* **21**, 1153 (2004), hep-th/0310234
55. Witten, E.: *Commun. Math. Phys.* **144**, 189 (1992)
56. Carlip, S.: *Phys. Rev. Lett.* **82**, 2828 (1999), hep-th/9812013
57. Carlip, S.: *Class. Quantum Gravity* **16**, 3327 (1999), gr-qc/9906126
58. Navarro, D.J., Navarro-Salas, J., Navarro, P.: *Nucl. Phys. B* **580**, 311 (2000), hep-th/9911091
59. Jing, J., Yan, M.-L.: *Phys. Rev. D* **63**, 024003 (2001), gr-qc/0005105

60. Izquierdo, J.M., Navarro-Salas, J., Navarro, P.: *Class. Quantum Gravity* **19**, 563 (2002), hep-th/0107132
61. Park, M.-I.: *Nucl. Phys. B* **634**, 339 (2002), hep-th/0111224
62. Silva, S.: *Class. Quantum Gravity* **19**, 3947 (2002), hep-th/0204179
63. Cvitan, M., Pallua, S., Prester, P.: *Phys. Lett. B* **555**, 248 (2003), hep-th/0212029
64. Cvitan, M., Pallua, S., Prester, P.: *Phys. Rev. D* **70**, 084043 (2004), hep-th/0406186
65. Cadoni, M.: hep-th/0511103
66. Dreyer, O., Ghosh, A., Wisniewski, J.: *Class. Quantum Gravity* **18**, 1929 (2001), hep-th/0101117
67. Koga, J.: *Phys. Rev. D* **64**, 124012 (2001), gr-qc/0107096
68. Pinamonti, N., Vanzo, L.: *Phys. Rev. D* **69**, 084012 (2004), hep-th/0312065
69. Huang, C.-G., Guo, H.-Y., Wu, X.-N.: *Phys. Rev. D* **68**, 104006 (2003), gr-qc/0304037
70. Solodukhin, S.N.: *Phys. Lett. B* **454**, 213 (1999), hep-th/9812056
71. Carlip, S.: *Phys. Rev. Lett.* **88**, 241301 (2002), gr-qc/0203001
72. Giacomini, A., Pinamonti, N.: *J. High Energy Phys.* **0302**, 014 (2003), gr-qc/0301038
73. Solodukhin, S.N.: *Phys. Rev. Lett.* **92**, 061302 (2004), hep-th/0310012
74. Ashtekar, A., Beetle, C., Fairhurst, S.: *Class. Quantum Gravity* **16**, L1 (1999), gr-qc/9812065
75. Booth, I.: gr-qc/0508107
76. Dirac, P.A.M.: *Can. J. Math.* **2**, 129 (1950)
77. Dirac, P.A.M.: *Can. J. Math.* **3**, 1 (1951)
78. Bergmann, P.G., Komar, A.B.: *Phys. Rev. Lett.* **4**, 432 (1960)
79. Carlip, S.: *Class. Quantum Gravity* **22**, 1303 (2005), hep-th/0408123
80. Carlip, S.: In: Scott, S., et al. (eds.) *The Kerr Spacetime: Rotating Black Holes in General Relativity*, Cambridge University Press, Cambridge (2006, in press), gr-qc/0508071
81. Louis-Martinez, D., Kunstatter, G.: *Phys. Rev. D* **52**, 3494 (1995), gr-qc/9503016
82. Grumiller, D., Kummer, W., Vassilevich, D.V.: *Phys. Rep.* **369**, 327 (2002), hep-th/0204253
83. Regge, T., Teitelboim, C.: *Ann. Phys.* **88**, 286 (1974)
84. Carlip, S.: gr-qc/0702107
85. Carlip, S., Teitelboim, C.: *Class. Quantum Gravity* **12**, 1699 (1995), gr-qc/9312002
86. Bergamin, L., Grumiller, D., Kummer, W., Vassilevich, D.V.: hep-th/0512230
87. Susskind, L., Thorlacius, L., Uglum, J.: *Phys. Rev. D* **48**, 3743 (1993), hep-th/9306069
88. Skenderis, K.: *Lect. Notes Phys.* **541**, 325 (2000), hep-th/9901050
89. Frolov, V.P., Fursaev, D., Zelnikov, A.: *J. High Energy Phys.* **0303**, 038 (2003), hep-th/0302207
90. Emparan, R., Sachs, I.: *Phys. Rev. Lett.* **81**, 2408 (1998), hep-th/9806122
91. Robinson, S.P., Wilczek, F.: *Phys. Rev. Lett.* **95**, 011303 (2005), gr-qc/0502074
92. Mattingly, D.: gr-qc/0601044